# Cosine 36° Problem
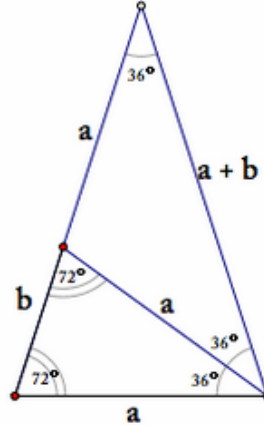
**Problem:**

a) Express $\cos\left(\frac{\pi}{5}\right)$ in closed form.

b) Prove that $4\cos\left(\frac{\pi}{5}\right)\cos\left(\frac{3\pi}{5}\right) = -1$.

**Solution:**

a) From Sublime Triangle, $\cos\left(\frac{\pi}{5}\right) = \frac{1+\sqrt{5}}{4}$



b) From Sublime Triangle, we know $\cos\left(\frac{\pi}{5}\right) = \frac{1+\sqrt{5}}{4} = \frac{1}{2}\cdot\frac{1+\sqrt{5}}{2} = \frac{\varphi}{2}$

Let $\frac{\pi}{5} = A$

Now, $\cos(3A) = 4\cos^3 A - 3\cos A = 4\left(\frac{\varphi}{2}\right)^3 - 3\left(\frac{\varphi}{2}\right) = \frac{\varphi^3}{2} - \frac{3\varphi}{2} = \frac{\varphi^3 - 3\varphi}{2}$

Furthermore, $\varphi^2 = \varphi + 1$

Putting it altogether,

$$4\cos\left(\frac{\pi}{5}\right)\cos\left(\frac{3\pi}{5}\right) = 4\cdot\frac{\varphi}{2}\cdot\frac{\varphi^3 - 3\varphi}{2}$$
$$= \varphi^4 - 3\varphi^2$$
$$= (\varphi + 1)^2 - 3(\varphi + 1)$$
$$= \varphi^2 + 2\varphi + 1 - 3\varphi - 3$$
$$= \varphi + 1 - \varphi - 2$$
$$= -1$$